

# A WILD MODEL OF LINEAR ARITHMETIC AND DISCRETELY ORDERED MODULES

PETR GLIVICKÝ AND PAVEL PUDLÁK

**ABSTRACT.** Linear arithmetics are extensions of Presburger arithmetic (Pr) by one or more unary functions, each intended as multiplication by a fixed element (scalar), and containing the full induction schemes for their respective languages.

In this paper we construct a model  $\mathcal{M}$  of the 2-linear arithmetic  $\text{LA}_2$  (linear arithmetic with two scalars) in which an infinitely long initial segment of “Peano multiplication” on  $\mathcal{M}$  is  $\emptyset$ -definable. This shows, in particular, that  $\text{LA}_2$  is not model complete in contrast to theories  $\text{LA}_1$  and  $\text{LA}_0 = \text{Pr}$  that are known to satisfy quantifier elimination up to disjunctions of primitive positive formulas.

As an application, we show that  $\mathcal{M}$ , as a discretely ordered module over the discretely ordered ring generated by the two scalars, is not NIP, answering negatively a question of Chernikov and Hils.

## 1. INTRODUCTION

There is longstanding interest in definability and related properties of various extensions of Presburger arithmetic ( $\text{Pr} = \text{Th}(\langle \mathbb{N}, 0, 1, +, \leq \rangle)$ ) by fragments of multiplication (see [Bès02] for a good survey). One class of such extensions are linear arithmetics, introduced in [Gli13] (but various similar situations were studied much earlier, see [Glib] for details). For any cardinal number  $\kappa$ , the  $\kappa$ -linear arithmetic  $\text{LA}_\kappa$  is an arithmetical theory containing the full induction scheme for its language  $\langle 0, 1, +, \leq, a_\alpha \rangle_{\alpha \in \kappa}$ , where each  $a_\alpha$  is a unary function symbol intended (and axiomatized) as multiplication by one fixed element (for the precise definition, see Section 2.1).

The theory  $\text{LA}_0$  is just Pr. Its definability properties are well understood. In particular, every formula is in Pr equivalent to a disjunction of bounded primitive positive formulas (bounded pp-formulas; i.e. formulas of the form  $(\exists \bar{x} < \bar{t})\chi(\bar{x}, \bar{y})$ , where  $\chi$  is a conjunction of atomic formulas), hence  $\text{LA}_0$  is model complete. Also it is a decidable theory. The same properties – bounded pp-elimination ([Gli13] or [Glib]), (consequently) model completeness and decidability ([Pen71] and independently [Gli13] or [Glib]) – have been shown also for  $\text{LA}_1$ . For  $\kappa \geq$

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2, nevertheless,  $\text{LA}_\kappa$  was only known to satisfy quantifier elimination up to bounded formulas [Gli1]. For more results on model theory of linear arithmetics, see [Glib] and [Gli1].

In this paper, we prove that the theories  $\text{LA}_\kappa$  with  $\kappa \geq 2$  are not model complete. We do this by constructing a model  $\mathcal{M} = \langle M, 0, 1, +, \leq, a, b \rangle \models \text{LA}_2$  such that for some  $L \in M$  nonstandard, an operation of partial Peano multiplication  $\cdot : [0, L]^2 \rightarrow M$  is  $\emptyset$ -definable in  $\mathcal{M}$  (see Theorem 3.1). Here,  $\emptyset$ -definable means definable without parameters and partial Peano multiplication is an operation  $\cdot$  that can be extended to the whole  $M^2$  in such a way that  $\langle M, 0, 1, +, \cdot, \leq \rangle$  is a model of Peano arithmetic (PA).

Note that, due to the bounded quantifier elimination in  $\text{LA}_\kappa$ , in no model  $\mathcal{M}$  of  $\text{LA}_\kappa$  Peano multiplication is definable on the whole  $M^2$ .

As an application of the above result, we show in section 4 that the constructed model  $\mathcal{M} \models \text{LA}_2$  endowed with a natural structure of a (discretely) ordered module has the independence property. This answers negatively the question of Chernikov and Hils [CH14, Question 5.9.1] whether all ordered modules are NIP.

Let us note that  $\text{LA}_1$  (as well as  $\text{LA}_0 = \text{Pr}$ ) is NIP, which follows easily from the quantifier elimination results in [Gli13], see [Glib].

Finally, let us remark that an analogous problem of definability of multiplication in expansions of the structure  $\langle \mathbb{R}, 0, +, < \rangle$  has been studied and that it exhibits surprisingly similar features as the problem of definability of multiplication in linear arithmetics (in particular, continued fractions are used in both cases). See [HT14] and [Hie16] for results formally closest to those in this paper.

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## 2. PRELIMINARIES

**2.1. Linear arithmetics.** For any cardinal number  $\kappa$ ,  $\kappa$ -linear arithmetic  $\text{LA}_\kappa$  is the theory in the language  $L_\kappa^{\text{lin}} = \langle 0, 1, +, \leq, a_\alpha \rangle_{\alpha \in \kappa}$  (where  $a_\alpha$  are unary function symbols) with the following axioms:

- (A1)  $0 \neq z + 1$ , (A2)  $x + 1 = y + 1 \rightarrow x = y$ ,
- (A3)  $x + 0 = x$ , (A4)  $x + (y + 1) = (x + y) + 1$ ,
- (D $_{\leq}$ )  $x \leq y \leftrightarrow (\exists z)(x + z = y)$ ,
- (L1)  $a_\alpha(x + 1) = a_\alpha x + a_\alpha 1$ , (L2)  $a_\alpha(a_\beta x) = a_\beta(a_\alpha x)$ ,
- (Ind)  $\varphi(0, \overline{y}) \& (\varphi(x, \overline{y}) \rightarrow \varphi(x + 1, \overline{y})) \rightarrow (\forall x)\varphi(x, \overline{y})$  for all formulas  $\varphi(x, \overline{y})$ .

Strictly  $\kappa$ -linear arithmetic  $\text{LA}_\kappa^\#$  is the extension of  $\text{LA}_\kappa$  by the axiomatic scheme

- (L $^\#$ ) “ $a_\alpha$  is not definable by any formula not containing  $a_\alpha$ ”.

**2.2. Models of  $\text{LA}_\kappa$  as discretely ordered modules.** Every  $\mathcal{M} \models \text{LA}_\kappa$  naturally corresponds to a discretely ordered module over a ring  $R$  given by  $\{a_\alpha; \alpha \in \kappa\}$ . Thus models of linear arithmetics can be understood as certain (in particular satisfying induction) discretely ordered modules. When  $\mathcal{M}$  is viewed in this way, we call the elements of the ring  $R$  scalars. Below we describe this correspondence in more detail.

$\text{LA}_\kappa$  proves that all elements are non-negative, but given a model  $\mathcal{M} \models \text{LA}_\kappa$ , it is often useful to formally add negative elements to  $\mathcal{M}$  and to work with this extension rather than with  $\mathcal{M}$  itself. In the rest of this paper we will not explicitly distinguish between these two structures and denote both simply by  $\mathcal{M}$ . This should not cause any confusion as the correct interpretation will always be clear from the context.

In every  $\mathcal{M} \models \text{LA}_\kappa$ , multiplication by any polynomial  $p \in \mathbb{Z}[a_\alpha]_{\alpha \in \kappa}$  can be naturally defined. Thus  $\mathcal{M}$  can be equipped with a structure of an (unordered)  $\mathbb{Z}[a_\alpha]_{\alpha \in \kappa}$ -module. It can be also viewed as an ordered module over the ordered ring

$$\mathbb{Z}(\mathcal{M}) := \mathbb{Z}[a_\alpha]_{\alpha \in \kappa} / \text{Ann}_{\mathbb{Z}[a_\alpha]_{\alpha \in \kappa}}(M),$$

where  $\text{Ann}$  denotes annihilator and the ordering of  $\mathbb{Z}(\mathcal{M})$  is induced by the ordering of  $\mathcal{M}$  via the map  $[p] \mapsto p1$ .

Let us note that, by induction in  $\mathcal{M}$ ,  $\text{Ann}_{\mathbb{Z}[a_\alpha]_{\alpha \in \kappa}}(M) = \text{Ann}_{\mathbb{Z}[a_\alpha]_{\alpha \in \kappa}}(\{1\})$ . Therefore  $\mathbb{Z}(\mathcal{M}) = \mathbb{Z}[a_\alpha]_{\alpha \in \kappa}$  (i.e.  $\mathcal{M}$  is a faithful  $\mathbb{Z}[a_\alpha]_{\alpha \in \kappa}$ -module) if and only if all  $a_\alpha$ 's are algebraically independent over  $\mathbb{Z}$  in  $\mathcal{M}$ .

Notice that if  $\mathcal{M}$  has the structure of an  $R$ -module (for any ring  $R$ ), then the map  $r \mapsto r1$  is a homomorphism from  $R$  (as a module over itself) to  $\mathcal{M}$ . We will often identify the scalar  $r$  with the element  $r1 \in M$ .

**2.3. Euclidean algorithm and continued fractions.** The proof of our main result is based on certain elementary properties of continued fractions that are provable in Peano arithmetic. We review all what is necessary here. A more detailed exposition can be found in [Khi97, Chapters I and II].

Let  $\mathcal{M}$  be a model of Peano arithmetic (PA). In this subsection, we will work in  $\mathcal{M}$ . This means that all quantifications, unless explicitly stated otherwise, are restricted to  $M$ . In the calculations, however, we will use negative elements and fractions freely.

Let us fix  $0 < b < a \in M$ . The Euclidean algorithm starting from  $(a, b)$  produces the division chain

$$r_{i-2} = r_{i-1}a_i + r_i, \tag{1}$$

for  $i = 0, \dots, n$ ,  $n \in M$ , where  $r_{-2} = a$ ,  $r_{-1} = b > r_0 > r_1 > \dots > r_n = 0$ ,  $r_{n-1} = \gcd(a, b)$ , and

$$[a_0; a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_n}}} = \frac{a}{b}$$

is the continued fraction of  $\frac{a}{b}$ .

The numerators and denominators of the convergents  $\frac{v_i}{u_i} = [a_0; \dots, a_i]$  (in the lowest terms) satisfy the recursive relations

$$u_i = u_{i-1}a_i + u_{i-2}; \quad v_i = v_{i-1}a_i + v_{i-2}, \quad (2)$$

for  $i = 0, \dots, n$ , where we set  $u_{-1} = v_{-2} = 0$  and  $u_{-2} = v_{-1} = 1$  (see [Khi97, Theorem 1] for a proof). Clearly,  $0 < u_0 < u_1 < \dots < u_n = b$  and  $0 \leq v_0 < v_1 < \dots < v_n = a$ .

From (1) and (2) it follows

$$r_i = (-1)^i (au_i - bv_i), \quad (3)$$

for  $i = -2, \dots, n$ , which can be rewritten as

$$\frac{a}{b} - \frac{v_i}{u_i} = (-1)^i \frac{r_i}{bu_i}.$$

From the last equation we can conclude that

$$\frac{v_0}{u_0} < \frac{v_2}{u_2} < \dots < \frac{v_n}{u_n} = \frac{a}{b} < \dots < \frac{v_3}{u_3} < \frac{v_1}{u_1}$$

and that  $|\frac{a}{b} - \frac{v_i}{u_i}|$  is decreasing (see also [Khi97, Theorem 4]).

Not only the convergents approximate  $\frac{a}{b}$  in this sense, they are exactly all the “best rational approximations of  $\frac{a}{b}$  of the second kind”, i.e. the following holds true (provably in PA):

**Proposition 2.1.** (see e.g. [Khi97, Theorems 16 and 17])

For  $u, v \in M$ ,  $u > 0$ , the following are equivalent:

- 1)  $|au - bv| < |au' - bv'|$  for all  $\frac{v'}{u'} \neq \frac{v}{u}$  with  $0 < u' \leq u$ .
- 2) There is  $0 \leq i \leq n$  such that  $\frac{v}{u} = \frac{v_i}{u_i}$ .

with the exception of  $\frac{a}{b} = a_0 + \frac{1}{2}$ , for which only 1)  $\Rightarrow$  2).

The above proposition gives a definition of the set of all convergents  $\frac{v_i}{u_i}$  of  $\frac{a}{b}$  by a bounded formula that uses multiplications only by  $a$  and  $b$ . We want a similar definition of the set of pairs  $(u_i, v_i)$ . Therefore we prove:

**Corollary 2.2.** Assume  $\frac{a}{b} \neq a_0 + \frac{1}{2}$ . Then for  $u, v \in M$ ,  $u > 0$ , the following are equivalent:

- 0\*)  $|au - bv| < |au' - bv'|$  for all  $(v', u') \neq (v, u)$  with  $0 < u' \leq u, b$  and  $0 \leq v' \leq a$ .
- 1\*)  $|au - bv| < |au' - bv'|$  for all  $(v', u') \neq (v, u)$  with  $0 < u' \leq u$ .
- 2\*) There is  $0 \leq i \leq n$  such that  $v = v_i$  and  $u = u_i$ .

*Proof.* “1\*)  $\Leftrightarrow$  2\*)”: We have the following chain of equivalences

$$1^* \Leftrightarrow 1 \ \& \ u, v \text{ are coprime} \Leftrightarrow 2 \ \& \ u, v \text{ are coprime} \Leftrightarrow 2^*,$$

where the second equivalence is from Proposition 2.1, and the other two are trivial.

“1\*)  $\Rightarrow$  0\*)” is obvious.

“0\*)  $\Rightarrow$  1\*)”: Let  $(v', u') \neq (v, u)$  and  $0 < u' \leq u$ . We prove  $|au - bv| < |au' - bv'|$ . First observe that for  $(v', u') = (a_0, 1)$ , we get from 0\*)

$$|au - bv| < |a1 - ba_0| = r_0 < b. \quad (4)$$

We may assume  $v' > 0$  (otherwise  $|au' - bv'| = au' \geq a > b$ , and we are done due to (4)). Further, let  $v' = ka + v''$ ,  $u' = lb + u''$  with  $0 < v'' \leq a$ ,  $0 < u'' \leq b$ ,  $k, l \in M$  (notice also that  $u'' \leq u' \leq u$ ). Then  $|au' - bv'| = |au'' - bv'' + (l - k)ab|$ . We distinguish the following cases:

- $l - k = 0$ : Then  $|au'' - bv'' + (l - k)ab| = |au'' - bv''| > |au - bv|$  by  $0^*$ .
- $l - k \geq 1$ : Then  $|au'' - bv'' + (l - k)ab| \geq a > b > |au - bv|$ , where the last inequality is due to (4) and the first one due to  $au'' - bv'' + (l - k)ab \geq a - ba + (l - k)ab \geq a$ .

□

### 3. WILD MODELS OF LINEAR ARITHMETICS

In this section we will construct a model of the arithmetic  $\text{LA}_2$  in which an infinite initial segment of a Peano multiplication is definable without parameters. In fact, we will prove even a little bit more. Say that a *formula is b-bounded* if all quantifiers in the formula are of the form  $\exists x < b1$ ,  $\forall x < b1$ . For the sake of simplicity, in this subsection we will write  $x < b$  instead of  $x < b1$  in the quantifier bounds.

**Theorem 3.1.** *Let  $\mathcal{M}$  be a non-standard model of PA and  $L \in M$ . Let  $\mathcal{M}^+$  be the additive part of  $\mathcal{M}$  and  $\cdot$  the operation of multiplication in  $\mathcal{M}$ . Then there are elements  $a < b \in M$  such that  $\cdot \upharpoonright [0, L]^2$  is  $\emptyset$ -definable by a b-bounded formula in  $\langle \mathcal{M}^+, a, b \rangle$ , where  $a, b$  stand for unary functions of multiplication by elements  $a, b$ .*

Note that if  $L$  is non-standard, then  $\langle \mathcal{M}^+, a, b \rangle \models \text{LA}_2^\#$  follows automatically for any  $a, b$  which satisfy the rest of the statement. Indeed, if one of the scalars were definable from the other (say  $b$  from  $a$ ) then the multiplication on  $[0, L]^2$  would be definable in  $\langle \mathcal{M}^+, a \rangle \models \text{LA}_1$ , which contradicts the pp-elimination in  $\text{LA}_1$ . (For any pp-formula  $\varphi(\bar{x})$  which defines an infinite set, it is easy to find  $\bar{u} \neq \bar{v}$  such that  $\varphi$  holds for  $\bar{u}, \bar{v}$  and  $\frac{\bar{u} + \bar{v}}{2}$ . But for  $\varphi(x, y, z)$  defining the graph of multiplication over the diagonal of  $[0, L]^2$ , this is clearly impossible.)

We will prove Theorem 3.1 in two steps. First, in subsection 3.1, we find three elements  $a, b, c \in M$  such that  $\cdot \upharpoonright [0, L]^2$  is  $\emptyset$ -definable in  $\langle \mathcal{M}^+, a, b, c \rangle \models \text{LA}_3$  ( $a, b, c$  here again standing for scalar multiplication by elements  $a, b, c$ ). Then, in subsection 3.2, we show that we can equivalently replace the scalars  $a, b, c$  in the definition of the multiplication by certain functions definable from only two scalars  $a', b'$ , and thus obtain a  $\emptyset$ -definition of  $\cdot \upharpoonright [0, L]^2$  in  $\langle \mathcal{M}^+, a', b' \rangle \models \text{LA}_2$ .

**3.1. A wild model of  $\text{LA}_3$ .** Let a non-standard  $\mathcal{M} = \langle \mathcal{M}^+, \cdot \rangle \models \text{PA}$  and  $L \in M$  be arbitrary. We will find elements  $a, b, c \in \mathcal{M}$  such that  $\cdot \upharpoonright [0, L]^2$  is  $\emptyset$ -definable in the extension  $\langle \mathcal{M}^+, a, b, c \rangle \models \text{LA}_3$  of  $\mathcal{M}^+$  by scalar multiplication by  $a, b, c$ . Note that any choice of  $a, b, c$  yields a model of  $\text{LA}_3$ , so we only need to take care about definability of the multiplication.

Let us explain the idea of our construction before going to the details. First, we can make things a bit easier by recalling the well-known fact that it is enough to define in  $\langle \mathcal{M}^+, a, b, c \rangle$  the function  $x \mapsto x^2$  on the domain  $[0, 2L]$ . Given the squaring function, multiplication on  $[0, L]^2$  is defined by

$$x \cdot y = \frac{(x + y)^2 - x^2 - y^2}{2}. \quad (5)$$

Let  $(z_i)_{i=0}^{4L-1} = (1, 1^2, 2, 2^2, \dots, 2L, (2L)^2)$  represent the required initial segment of  $x \mapsto x^2$ . The idea is then to pick the scalar  $c$  as any prime number bigger than  $(2L)^2$  and define  $a$  and  $b$  in such a way that the numerators  $v_i$ ,  $i < 4L$ , of convergents of the continued fraction  $[a_0; \dots, a_{4L-1}]$  representing  $a/b$  encode the initial segment of  $x \mapsto x^2$  in the following sense:

$$z_i = v_i \bmod c, \quad (6)$$

for  $i = 0, \dots, 4L - 1$ .

By Corollary 2.2, the set  $\{v_i; i < 4L\}$  is  $\emptyset$ -definable in  $\langle \mathcal{M}^+, a, b, c \rangle$  (in fact in  $\langle \mathcal{M}^+, a, b \rangle$ ). Then combining this definition with (5) and (6) easily gives the sought definition of multiplication on  $[0, L]^2$ .

Now we describe the construction in detail. As mentioned above, we choose  $c$  to be any prime bigger than  $(2L)^2$ . In order to define  $a$  and  $b$ , we recursively choose the coefficients  $a_i$  of the continued fraction  $[a_0; \dots, a_{4L-1}]$  in such a way that (6) holds true for every  $0 \leq i < 4L$  with the numerators  $v_i$  computed from  $a_i$ s using (2). Then we take  $a$  and  $b$  coprime such that  $a/b = [a_0; \dots, a_{4L-1}]$ .

Let  $0 \leq i < 4L$  and suppose that we have already defined  $a_j$  for all  $0 \leq j < i$  in such a way that (6) holds. Notice that no  $v_j$  with  $-1 \leq j < i$  is divisible by  $c$ , since  $v_{-1} = 1$  and for  $j \geq 0$ ,  $v_j \equiv z_j \bmod c$  and  $0 < z_j \leq (2L)^2 < c$ . Therefore there is  $a_i > 0$  such that

$$z_i \equiv v_i = v_{i-1}a_i + v_{i-2} \bmod c,$$

i.e. (6) holds for  $i$ .

It remains to show that with  $a, b, c$  defined in this way, we can find an  $L_3^{lin}$ -formula which defines  $x \mapsto x^2$  on  $[0, 2L]$  in  $\langle \mathcal{M}^+, a, b, c \rangle$ .

Let  $\gamma(u, v)$  be the  $L_3^{lin}$ -formula

$$\gamma(u, v): (\forall u', 0 < u' \leq u)(\forall v', 0 \leq v' \leq a)((u, v) \neq (u', v') \rightarrow |au - bv| < |au' - bv'|).$$

(This is  $0^*$  from Corollary 2.2 without the bound  $u' \leq b$ .) Then the  $L_3^{lin}$ -formulas

$$V(v): (\exists u, 0 < u \leq b)\gamma(u, v), \quad (7)$$

$$V_0(v): (\exists u, 0 < u \leq b)(\gamma(u, v) \& au - bv > 0), \quad (8)$$

$$V_1(v): (\exists u, 0 < u \leq b)(\gamma(u, v) \& au - bv \leq 0), \quad (9)$$

define the sets  $V = \{v_i; 0 \leq i < 4L\}$ ,  $V_0 = \{v_i; 0 \leq i < 4L \text{ and } i \text{ even}\}$  and  $V_1 = \{v_i; 0 \leq i < 4L \text{ and } i \text{ odd}\}$  respectively in  $\langle \mathcal{M}^+, a, b, c \rangle$ . For  $V$ , this follows directly from Corollary 2.2. The cases of  $V_0$  and  $V_1$  are implied by (3) with the case  $au - bv = 0$  falling into  $V_1$ , as this is only possible for  $i = 4L - 1$  which is odd.

Notice also that since the sequence  $(v_i)$  is increasing, we can define the set of all pairs  $(v_{2i}, v_{2i+1})$  with  $i < 2L$  by:

$$\pi(v, v'): V_0(v) \& V_1(v') \& \neg(\exists w, v < w < v')V(w).$$

Finally, we define  $x \mapsto x^2$  on  $[0, 2L]$  by:

$$x^2 = y \leftrightarrow x = y = 0 \vee (\exists v, v')(0 \leq v, v' \leq a \& \pi(v, v') \& x = v \bmod c \& y = v' \bmod c),$$

where, of course,  $z = w \bmod c$  stands for  $0 \leq z < c \& (\exists m)(0 \leq m \leq w \& w = z + cm)$ .

We denote the formula on the right hand side of the previous equivalence by  $\sigma(x, y)$ . Notice that this is a bounded formula (even bounded by constant terms) and does not contain parameters from  $M$ . The same holds true about the formula which defines multiplication on  $[0, L]^2$  using (5), as it is equivalent to

$$\mu(x, y, z): (\exists p, q, r)(0 \leq p, q, r < c \& \sigma(x, p) \& \sigma(y, q) \& \sigma(x + y, r) \& 2z + p + q = r),$$

where all three numbers  $r = (x + y)^2$ ,  $p = x^2$ ,  $q = y^2$  can be bounded by  $(2L)^2$  and therefore by  $c$ .

**Remark 3.2.** *Note that in the above construction we did not use any specific property of the constructed squaring function besides that it is nonzero, its range is bounded in  $\mathcal{M}$  and that it is coded in  $\mathcal{M}$  (via Gödel's coding of finite sets). A slight modification of this construction could be therefore used to yield a nonstandard segment of any unary function  $f$  (or, with only little more modifications, even any  $n$ -ary relation  $R$  for arbitrary  $n$ ) on  $M$  that is coded in  $\mathcal{M}$ . (The requirement of being nonzero can be easily overcome by constructing  $f + 1$  instead of  $f$  and subtracting 1 at the end, and of course any coded relation is bounded in all coordinates in  $M$ .)*

**3.2. A wild model of  $\text{LA}_2$ .** Let  $\langle \mathcal{M}^+, a, b, c \rangle$  be the model from the previous subsection. We will show that the multiplication on  $[0, L]^2$  is  $\emptyset$ -definable in the structure  $\langle \mathcal{M}^+, ac, abc^2 + c \rangle \models \text{LA}_2$  (where again  $ac$  and  $abc^2 + c$  stand for the functions of scalar multiplication by these two elements). If we could prove that scalar multiplication by  $a$ ,  $b$  and  $c$  is definable using scalar multiplication by  $ac$  and  $abc^2 + c$ , we would be done, but this is not the case. We are only able to define the elements  $a1$ ,  $b1$  and  $c1$ . We would also be done, if we could define scalar multiplication by  $ac$ ,  $bc$  and  $c$ , because the formula defining partial multiplication is homogeneous in  $a$  and  $b$ . We do have  $ac$ , but we are not able to define  $bc$  and  $c$ . However, what we can do is to define scalar multiplication by  $bc$  and  $c$  restricted to the interval  $[0, a1]$ , which suffices for our purpose.

Let  $\alpha^*x := acx$  for all  $x$ ,  $\beta^*x = bcx$  for  $0 \leq x \leq a$ ,  $\gamma^*x = cx$  for  $0 \leq x \leq a$  and let  $\gamma^*x = \beta^*x = 0$  for  $x > a$ . We will modify the formula  $\mu$  defined above as follows. We keep  $a1$ ,  $b1$  and  $c1$  in the inequalities that determine the range of quantification. (In the formula we used just letters  $a$ ,  $b$  and  $c$  for the sake of brevity; now we have to be more careful.) We replace the other occurrences of scalar multiplication by  $a$ ,  $b$  and  $c$  by the functions  $\alpha^*$ ,  $\beta^*$  and  $\gamma^*$  respectively. We will denote the resulting formula by  $\mu'(x, y, z)$ .

First we prove that  $\mu(x, y, z) \Leftrightarrow \mu'(x, y, z)$  for all  $x, y, z \in M$ . In what follows, for a subformula  $\varphi$  of  $\mu$ , we denote by  $\varphi'$  the corresponding subformula of  $\mu'$ .

We observe that during the evaluation of  $\mu(x, y, z)$ , the subformulas  $V(v)$ ,  $V_0(v)$  and  $V_1(v)$  are only evaluated for  $0 \leq v \leq a$ . For  $0 \leq v, v' \leq a$  (and any  $u, u'$ ), we have

$$|au - bv| < |au' - bv'| \Leftrightarrow |\alpha^*u - \beta^*v| < |\alpha^*u' - \beta^*v'|$$

and similarly for  $au - bv > 0$  and  $au - bv \leq 0$ . Therefore, for  $0 \leq v \leq a$ , we get

$$V(v) \Leftrightarrow V'(v)$$

and the same for  $V'_0$ ,  $V'_1$ . Consequently, for  $0 \leq u, v \leq a$ ,

$$\pi(u, v) \Leftrightarrow \pi'(u, v).$$

Further, for  $0 \leq w \leq a$  and any  $z$ , we get that

$$z = w \bmod c \Leftrightarrow (z = w \bmod c)'. \quad (1)$$

(Note that  $z = w \bmod c$  means  $z \equiv w \bmod c$  &  $0 \leq z < c1$ .) From this, it is easy to see that the same equivalence holds true also for  $\sigma(x, y)$  and consequently for  $\mu(x, y, z)$ .

It remains to find definitions of  $\beta^*, \gamma^*, a1, b1, c1$  in  $\langle \mathcal{M}^+, ac, abc^2 + c \rangle$ . Let us denote by  $\alpha = ac, \beta = bc$ . Then  $abc^2 + c = \alpha\beta + c$ .

To define  $\gamma^*$ , we first define an auxiliary function  $\gamma^\circ$  by

$$\gamma^\circ x = ((\alpha\beta + c)x) \bmod \alpha.$$

Notice that for  $0 \leq x < a1$ ,  $\gamma^\circ x = cx = \gamma^*x$ , but  $\gamma^\circ a = 0 \neq \gamma^*a$ . This enables us to write down parameter-free definitions of  $a1, c1$  and  $\gamma^*$  in  $\langle \mathcal{M}^+, \alpha, \alpha\beta + c \rangle$ :

$$\begin{aligned} a1 &= \min\{x > 0; \gamma^\circ x = 0\}, \\ c1 &= \gamma^\circ 1, \end{aligned}$$

and

$$\gamma^*x = \begin{cases} \gamma^\circ x & \text{for } 0 \leq x < a1, \\ \alpha 1 & \text{for } x = a1, \\ 0 & \text{otherwise.} \end{cases}$$

To define  $\beta^*$ , we again start with a definition of an auxiliary function  $\beta^\circ$

$$\beta^\circ x = ((\alpha\beta + c)x) \operatorname{div} \alpha,$$

where the function  $u \operatorname{div} \alpha$  is defined by  $w = u \operatorname{div} \alpha \Leftrightarrow \alpha w \leq u < \alpha(w + 1)$ . Again, it is not difficult to see that  $\beta^\circ x = bcx = \beta^*x$  for  $0 \leq x < a$ , but  $\beta^\circ a = abc + 1 \neq \beta^*a$ , which we can use to  $\emptyset$ -define  $b1$  and  $\beta^*$  in  $\langle \mathcal{M}^+, \alpha, \alpha\beta + c \rangle$  as follows:

$$b1 = (\beta^\circ a) \operatorname{div} \alpha,$$

and

$$\beta^*x = \begin{cases} \beta^\circ x & \text{for } 0 \leq x < a1, \\ \beta^\circ x - 1 & \text{for } x = a1, \\ 0 & \text{otherwise.} \end{cases}$$

Finally note that the  $L_2^{lin}$ -definition  $\mu''(x, y, z)$  of multiplication on  $[0, L]^2$  in  $\langle \mathcal{M}^+, \alpha, \alpha\beta + c \rangle$  that we just constructed is a bounded formula, because it was constructed from  $\mu'$  by substituting definitions of functions  $\alpha^*, \beta^*, \gamma^*$  and constants  $a1, b1, c1$  to appropriate places. It can easily be seen that during the evaluation of  $\mu'$  in  $\langle \mathcal{M}^+, \alpha^*, \beta^*, \gamma^* \rangle$  the function  $\alpha^*x$  is evaluated only for  $0 \leq x \leq b$  and  $\beta^*x, \gamma^*x$  only for  $0 \leq x \leq a$ . Therefore, always  $0 \leq \alpha^*x, \beta^*x, \gamma^*x < abc1 <$



$(\alpha\beta + c)1$ . Clearly also  $0 \leq a1, b1, c1 < (\alpha\beta + c)1$  and thus the existentially quantified variables in  $\mu''$  can be bounded by  $(\alpha\beta + c)1$ .

#### 4. A NON-NIP DISCRETELY ORDERED MODULE

A structure  $\mathcal{A}$  is NIP (not independence property; see [Sim15] for an extensive introduction to NIP structures and theories) if there is no formula  $\varphi(\bar{x}, \bar{y})$  such that for every  $n \in \omega$ , there are  $\bar{a}_i \in A^{l(\bar{x})}$ , with  $i < n$ , and  $\bar{b}_J \in A^{l(\bar{y})}$ , with  $J \subseteq n$ , such that

$$\varphi(\bar{a}_i, \bar{b}_J) \Leftrightarrow i \in J.$$

Chernikov and Hils [CH14, Question 5.9.1] asked whether all ordered modules are NIP. We answer their question negatively:

Let  $\langle \mathcal{M}^+, a, b \rangle \models \text{LA}_2$  be a model in which a multiplication (function  $\cdot$  satisfying  $x \cdot 0 = 0$  and  $x(y + 1) = xy + x$ ) is definable on  $[0, L]^2$  for some non-standard  $L$  (such models exist by Theorem 3.1) and let  $\mathcal{A} = \langle M, 0, +, -, \leq, r \rangle_{r \in R}$  be the discretely ordered module corresponding to  $\langle \mathcal{M}^+, a, b \rangle$  (see subsection 2.2).

**Proposition 4.1.** *The discretely ordered module  $\mathcal{A}$  is not NIP.*

*Proof.* Let  $\psi$  define multiplication  $\cdot$  on  $[0, L]^2$  in  $\langle \mathcal{M}^+, a, b \rangle$ . We construct a formula  $\psi'$  by replacing possible occurrences of the constant 1 in  $\psi$  by the definition of 1 in  $\mathcal{A}$  (the least positive element), replacing every quantifier  $(Qx)$  by  $(Qx, 0 \leq x)$  and replacing scalars  $a, b$  by scalars  $q, r \in R$  representing the same functions on  $M$  as  $a, b$  do. Then the formula  $x, y, z \geq 0 \ \& \ \psi'(x, y, z)$  defines  $\cdot$  on  $[0, L]^2$  in  $\mathcal{A}$ .

Clearly  $\cdot \upharpoonright \mathbb{N}^2$  is the usual multiplication on  $\mathbb{N}$  and the formula  $\varphi(x, y) : (\exists z, 0 \leq z \leq y) z \cdot x = y$  when restricted to  $\mathbb{N}^2$  defines the usual divisibility relation.

It is now easy to prove that the  $\varphi$  has the independence property: Let  $n \in \omega$  be given. For  $i < n$  take  $a_i$  the  $i$ -th prime number in  $\mathbb{N}$  and for  $J \subseteq n$  take  $b_J = \prod_{i \in J} a_i$ .  $\square$

#### 5. OPEN PROBLEMS

What is the strongest possible quantifier elimination result for  $\text{LA}_\kappa$  with  $\kappa \geq 2$ ? Can definable sets in models of  $\text{LA}_\kappa$  be precisely characterized?

Is there a model of  $\text{LA}_\kappa^\#$  with  $\kappa \geq 2$  whose first-order theory is model complete/NIP? Can those models be characterized?

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PETR GLIVICKÝ: ACADEMY OF SCIENCES OF THE CZECH REPUBLIC, INSTITUTE OF MATHEMATICS, ŽITNÁ 25, 115 67 PRAHA 1, CZECH REPUBLIC

PETR GLIVICKÝ: UNIVERSITY OF ECONOMICS, DEPARTMENT OF MATHEMATICS, EKONOMICKÁ 957, 148 00 PRAHA 4, CZECH REPUBLIC

PAVEL PUDLÁK: ACADEMY OF SCIENCES OF THE CZECH REPUBLIC, INSTITUTE OF MATHEMATICS, ŽITNÁ 25, 115 67 PRAHA 1, CZECH REPUBLIC